

MEI Structured Mathematics

Practice Comprehension Question - 1

(Concepts for Advanced Mathematics, C2)

Pythagorean and other Triples

Pythagoras was born on the Greek island of Samos in BC572 and founded a school of philosophy, mathematics and natural science in the Greek seaport of Crotona in Southern Italy. He is, of course, famous for the theorem that bears his name, that in a right angled triangle the square of the hypotenuse is equal to the sum of the squares of the other two sides.



Fig. 1

$$a^2 + b^2 = c^2$$

$$3^2 + 4^2 = 5^2 \quad (9 + 16 = 25)$$

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A set of three integers which can form the sides of a right-angled triangle, such as {3,4,5} is called a Pythagorean Triple.

One way of generating Pythagorean Triples is as follows:

	Example	Algebraically
Think of an odd number	7	$2n + 1$
Square it	49	$4n^2 + 4n + 1$
Split your answer into two numbers, one smaller by one than the other	24,25	$2n^2 + 2n, 2n^2 + 2n + 1$
The number you first thought of and these two form a Pythagorean Triple.	{7, 24, 25}	$\{2n + 1, 2n^2 + 2n, 2n^2 + 2n + 1\}$

There are other Pythagorean Triples, such as {8, 15, 17} which cannot be generated by this method.

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It can, however, be shown that all Pythagorean Triples are generated by the formula

$$\{2mn, m^2 - n^2, m^2 + n^2\}$$

Where m and n are integers. For instance, substituting $m = 4$ and $n = 1$ gives the Triple {8, 15, 17}.

There are also triangles with one angle 60° and sides of integer length, like the two shown in Fig. 2 below and a generator for these triangles is

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$$\{p^2 + 2pq, 2pq + q^2, p^2 + pq + q^2\}$$

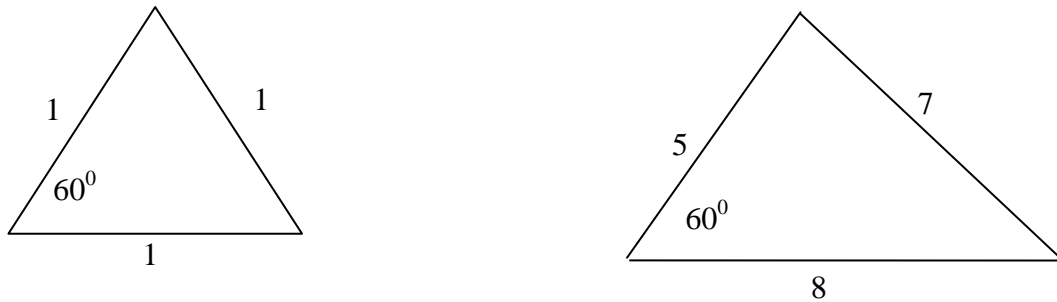


Fig. 2

Questions:

- 1** Pythagoras Theorem states:
In a right-angled triangle the sum of the squares of two sides is equal to the square on the hypotenuse.
State the converse of this theorem. [1]
- 2** In Table 1, the number 7 is used as a starting point in the example. Follow the procedure starting with the number 11 to produce a Pythagorean Triple. [2]
- 3** Explain why a general odd number has the form $2n + 1$. [1]
- 4** Prove that $\{2n + 1, 2n^2 + 2n, 2n^2 + 2n + 1\}$ is a pythagoran triangle. [2]
- 5** In line 12 the generator for Pythagorean Triples is given.
- (i) Show that this Triple is a Pythagorean Triangle for all positive integers m, n where you may take $m \geq n$. [2]
- (ii) Find the Triple generated by $m = 3$ and $n = 2$. [1]
- (iii) Find the values of m and n which generate the triple $\{140, 51, 149\}$. [2]
- (iv) Copy and continue the table below to generate 5 *different* triples.
(In this context $\{6, 8, 10\}$ is a multiple of $\{3, 4, 5\}$ and so is not different.) [2]

m	n	$2mn$	$m^2 - n^2$	$m^2 + n^2$
2	1	4	3	5
3	1	6	8	10
3	2			

- 6** (i) In Fig. 2 there is an example given with sides 5, 8 and 7 where an angle is 60° . Find the values of p and q that generate this Triple. [2]
- (ii) Prove that with the generator given in line 16, one angle is indeed 60° . [3]

Answers.

1	In a triangle, if the square of the longest side is equal to the sum of squares on the two shorter sides then the triangle is right-angled.	B1 1																																																							
2	11 121 65,66 giving { 11, 65, 66}	M1 A1 2																																																							
3	$2n$ must be even for all n because it has a factor of 2. therefore $2n + 1$ must be odd.	B1 1																																																							
4	$(2n+1)^2 + (2n^2 + 2n)^2$ $= 4n^2 + 4n + 1 + 4n^4 + 8n^3 + 4n^2$ $= 4n^4 + 8n^3 + 8n^2 + 4n + 1$ $(2n^2 + 2n + 1)^2 = 4n^4 + 4n^2(2n+1) + (2n+1)^2$ $= 4n^4 + 8n^3 + 4n^2 + 4n^2 + 4n + 1$ $= 4n^4 + 8n^3 + 8n^2 + 4n + 1$ So the sum of squares of two is equal to the square of the third.	M1 A1 2																																																							
5 (i)	$(2mn)^2 + (m^2 - n^2)^2 = 4m^2n^2 + m^4 - 2m^2n^2 + n^4$ $= m^4 + 2m^2n^2 + n^4 = (m^2 + n^2)^2$	M1 A1 2																																																							
(ii)	12, 5, 13	B1 1																																																							
(iii)	We must have $2mn = 140 \Rightarrow mn = 70$ $m^2 - n^2 = 51$ $m^2 + n^2 = 149$ Add: $2m^2 = 200 \Rightarrow m^2 = 100$ $\Rightarrow m = 10, \Rightarrow n = 7$ i.e. {7, 51, 149}	M1 A1 2																																																							
(iv)	<table border="1"> <thead> <tr> <th>m</th> <th>n</th> <th>$2mn$</th> <th>$m^2 - n^2$</th> <th>$m^2 + n^2$</th> <th></th> </tr> </thead> <tbody> <tr> <td>2</td> <td>1</td> <td>4</td> <td>3</td> <td>5</td> <td>No. 1</td> </tr> <tr> <td>3</td> <td>1</td> <td>6</td> <td>8</td> <td>10</td> <td>mult</td> </tr> <tr> <td>3</td> <td>2</td> <td>12</td> <td>5</td> <td>13</td> <td>No. 2</td> </tr> <tr> <td>4</td> <td>1</td> <td>8</td> <td>15</td> <td>17</td> <td>No. 3</td> </tr> <tr> <td>4</td> <td>2</td> <td></td> <td></td> <td></td> <td>Mult</td> </tr> <tr> <td>4</td> <td>3</td> <td>24</td> <td>7</td> <td>25</td> <td>No. 4</td> </tr> <tr> <td>5</td> <td>1</td> <td>10</td> <td>24</td> <td>26</td> <td>Mult.</td> </tr> <tr> <td>5</td> <td>2</td> <td>20</td> <td>21</td> <td>29</td> <td>No. 5</td> </tr> </tbody> </table>	m	n	$2mn$	$m^2 - n^2$	$m^2 + n^2$		2	1	4	3	5	No. 1	3	1	6	8	10	mult	3	2	12	5	13	No. 2	4	1	8	15	17	No. 3	4	2				Mult	4	3	24	7	25	No. 4	5	1	10	24	26	Mult.	5	2	20	21	29	No. 5	M1 A1 2	some sort of methodical method All 5
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4	3	24	7	25	No. 4																																																				
5	1	10	24	26	Mult.																																																				
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6	<p>(i) Either p or q must be equal to 1. This is because $p^2 + 2pq = p(p + 2q)$ and $2pq + q^2 = q(q + 2p)$ and one of them equals 5 which is prime. So take $p = 1$ giving $1 + 2q = 5 \Rightarrow q = 2$ $\{p^2 + 2pq, 2pq + q^2, p^2 + pq + q^2\}$</p>	M1 A1 2	
	<p>(ii) By cosine rule: $\cos A = \frac{(p^2 + 2pq)^2 + (q^2 + 2pq)^2 - (p^2 + pq + q^2)^2}{2(p^2 + 2pq)(q^2 + 2pq)}$ $= \frac{p^4 + 4p^3q + 4p^2q^2 + q^4 + 4pq^3 + 4p^2q^2 - p^4 - 2p^2(pq + q^2) - (pq + q^2)^2}{2(p^2 + 2pq)(q^2 + 2pq)}$ $= \frac{2p^3q + 2p^2q^2 + q^4 + 4pq^3 + 4p^2q^2 - p^2q^2 - 2pq^3 - q^4}{2(p^2 + 2pq)(q^2 + 2pq)}$ $= \frac{2p^3q + 5p^2q^2 + 2pq^3}{2(p^2 + 2pq)(q^2 + 2pq)} = \frac{(p^2 + 2pq)(q^2 + 2pq)}{2(p^2 + 2pq)(q^2 + 2pq)} = \frac{1}{2}$ $\Rightarrow A = 60^\circ$</p>	M1 A1 A1 3	