

MEI Structured Mathematics

Practice Comprehension Question - 2

(Concepts for Advanced Mathematics, C2)

Perfect Numbers

The factors of the number 24 are 1, 2, 3, 4, 6, 8, 12 and 24. The sum of these factors is found by adding them together:

$$1 + 2 + 3 + 4 + 6 + 12 + 24 = 60$$

Now divide this sum by the original number, 24 and you get $\frac{60}{24} = 2\frac{1}{2}$.

This article looks at how the answer to this process is related to the original number. 5

The notation to be used in this article is as follows:

Let n be the original number, which is a positive integer

Let s be the sum of all the factors of n

The k be the result of dividing s by n $k = \frac{s}{n}$

So in the example above, $n = 24$, $s = 60$ and $k = \frac{60}{24} = 2\frac{1}{2}$ 10

There will be a few numbers, such as $n = 1$ and $n = 6$ where k is an integer, though for $n = 24$, it is not.

If $k > 2$ then n is said to be **abundant**; 24 is an abundant number.

If $k < 2$ then n is said to be **deficient**; all prime numbers are deficient.

If $k = 2$ then n is said to be **perfect**; 6 is a perfect number.

You will realise that the value $k = 2$ comes about by including the number itself as a factor. 15

This is equivalent to the widely recognised definition:

*A perfect number is one in which the sum of all its factors **excluding the number itself** is equal to the number.*

Perfect numbers have long held a fascination for mathematicians. Euclid wrote about them over 2000 years ago. In times gone by magical properties were ascribed to them. 20

Some values of n , s and k are given in Table 1 below.

n	1	2	6	8
s	1	3	12	15
k	1	$1\frac{1}{2}$	2	$1\frac{7}{8}$

Table 1

The table could be extended *ad infinitum* to identify all perfect numbers. To find that the next perfect number is 28 may seem tedious but to discover that the next number giving an integer value of k is 120 would take a great deal of work. Writing down all the factors of a large number is tedious, and there is always the worry that you may have missed one or two out, leading to incorrect values for s and k and therefore an incorrect identification of whether the number is or is not perfect. 25

Clearly for all integers greater than 1, $s > n$ and so $k > 1$. For prime numbers $k = 1 + \frac{1}{p}$.

Fortunately it is possible to avoid listing (and adding) all the factors; there is an alternative way to discover whether or not a number is perfect. The method is described in the following examples. 30

Example 1: $n = 28$

Write down the prime factorisation: $28 = 2^2 \times 7$.

We generate the full set of factors of 28 (which are $\{1, 2, 4, 7, 14, 28\}$), by noting that the factors are all of the form $2^a \times 7^b$ where $a = 0, 1$ or 2 and $b = 0$ or 1 . 35

So the factors are $2^0 \times 7^0, 2^1 \times 7^0, 2^2 \times 7^0, 2^0 \times 7^1, 2^1 \times 7^1, 2^2 \times 7^1$

Finding the sum s , of these factors is the same as multiplying out the brackets

$$\begin{aligned} s &= (1 + 2 + 2^2) \times (1 + 7) \\ &= 7 \times 8 = 56 \end{aligned}$$

Therefore $k = \frac{s}{n} = \frac{56}{28} = 2$; 28 is perfect. 40

Example 2: $n = 24$.

$$24 = 2^3 \times 3$$

$$\Rightarrow s = (1 + 2 + 2^2 + 2^3) \times (1 + 3)$$

$$\Rightarrow s = 15 \times 4 = 60$$

[Note that expanding the brackets gives 45

$$\begin{aligned} s &= 1 + 2 + 2^2 + 2^3 + 1 \times 3 + 2 \times 3 + 2^2 \times 3 + 2^3 \times 3 \\ &= 1 + 2 + 4 + 8 + 3 + 6 + 12 + 24 = 60] \end{aligned}$$

$$k = \frac{s}{n} = \frac{60}{24} = 2.5; 24 \text{ is abundant.}$$

Example 3: $n = 945$

$$945 = 3^3 \times 5 \times 7$$

$$s = (1 + 3 + 3^2 + 3^3)(1 + 5)(1 + 7) = 40 \times 6 \times 8 = 1920$$

$$k = \frac{s}{n} = \frac{1920}{945} = 2.03\dots; 945 \text{ is also abundant.}$$

(Incidentally, 945 is the smallest odd abundant number.)

This gives us a way of determining whether a chosen number is perfect or not, but it does not give us a pattern for finding the numbers.

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A way of finding such numbers has been known for over 2000 years when it was described by Euclid.

It can be stated as

$$n = 2^{p-1}(2^p - 1) \text{ is perfect if } 2^p - 1 \text{ is prime.}$$

This can be proved as follows.

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If n has a prime factorisation of 2^{p-1} and a prime number of the form $(2^p - 1)$ then

- (i) the factors include all powers of 2 up to 2^{p-1} - i.e. 1, 2, 2^2 , 2^3 , ..., 2^{p-1}
- (ii) the factors of $(2^p - 1)$ are 1 and itself.

So write $s = (1 + 2 + 2^2 + 2^3 + \dots + 2^{p-1})(1 + 2^p - 1)$

$(1 + 2 + 2^2 + 2^3 + \dots + 2^{p-1})$ is a geometric progression with first term 1 and common ratio 2.

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$$(1 + 2 + 2^2 + 2^3 + \dots + 2^{p-1}) = \frac{2^p - 1}{2 - 1} = 2^p - 1$$

Thus $s = (2^p - 1)2^p$

and $k = \frac{s}{n} = \frac{(2^p - 1)2^p}{2^{p-1}(2^p - 1)} = 2$

And so n is perfect.

Table 2 below gives the values of $(2^p - 1)$ for $1 \leq p \leq 5$. Where it is prime the corresponding value for n is also given.

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p	$2^p - 1$	Prime?	n
1	1	No	
2	3	Yes	6
3	7	Yes	28
4	15	No	
5	31	Yes	496

Table 2

This sequence of perfect numbers is 6, 28, 496....

The terms rapidly become very large and it ceases to be a simple matter to decide whether or not $2^p - 1$ is prime.

Numbers of the form $2^p - 1$ are called *Mersenne numbers* in honour of the French Mathematician Marin Mersenne (1588 - 1648) who first investigated them seriously. His work included developing a short cut method for deciding if a Mersenne number is prime, making the search for large perfect numbers less arduous than it would otherwise be. Mersenne numbers, such as 3, 7 and 31, which are also prime are called *Mersenne primes*.
There are not many known perfect numbers; by 2004 only 41 had been found. The largest of these is $2^{240365582}(2^{24036583} - 1)$ which has 14 471 465 digits!

All known perfect numbers are of the form $2^{p-1} \times (2^p - 1)$ and so are even. No one has yet found a perfect number that is odd. However, to prove that no such number exists remains one of the unsolved problems of mathematics.

Questions:

- 1 (i) By listing all the factors, show that 6 is perfect. [1]
- (ii) Show that 10 is deficient. [2]
- (iii) In line 12 it says that when $k > 2$ the number is abundant. By filling in (and, if necessary, extending) Table 1, find the smallest abundant number. [2]
- 2 Prove the statement in line 24: For all prime numbers, p , $k = 1 + \frac{1}{p}$. [2]
- 3 Using the method of the examples on page 2, show that the value of k for 120 is an integer. [3]
- 4 Investigate the number 32760. [4]
- 5 Write down the general formula being used in lines 65 and 66. [1]
- 6 Given that $2^p - 1$ is prime for $p = 2, 3, 5, 7$ and 13 but for no other values of $p < 16$, find the first five perfect numbers. [3]

Answers.

1	(i)	Factors of 6 are 1, 2, 3 $1 + 2 + 3 = 6$	B1	1																																	
	(ii)	Factors of 10 are 1, 2, 5, 10 $1 + 2 + 5 + 10 = 18$ $\frac{18}{15} = 1.2 < 2.$	B1 B1	2																																	
	(iii)	Smallest abundant number is 12	M1 A1	2																																	
2		The factors of a prime number p are 1 and p . Therefore $s = 1 + p$. $\Rightarrow k = \frac{p+1}{p} = 1 + \frac{1}{p}$	M1 A1	2																																	
3		$120 = 2^3 \times 3 \times 5$ $\Rightarrow s = (1 + 2 + 2^2 + 2^3)(1 + 3)(1 + 5)$ $= \frac{2^4 - 1}{2 - 1} \times 4 \times 6 = 360$ $\Rightarrow k = \frac{360}{120} = 3$	B1 M1 A1	3																																	
4		$32760 = 2^3 \times 3^2 \times 5 \times 7 \times 13$ $\Rightarrow s = (1 + 2 + 2^2 + 2^3)(1 + 3 + 3^2)(1 + 5)(1 + 7)(1 + 13)$ $= \frac{2^4 - 1}{2 - 1} \times 13 \times 6 \times 8 \times 14 = 15 \times 13 \times 6 \times 8 \times 14$ $\Rightarrow k = \frac{15 \times 13 \times 6 \times 8 \times 14}{2^3 \times 3^2 \times 5 \times 7 \times 13} = 4$ So 32760 is a 4-perfect number.	B1 M1 A1 A1	4																																	
5		The general formula for a G.P. with first term a and common ratio r is $S_n = a + ar + ar^2 + \dots + ar^{n-1}$ $= \frac{a(r^n - 1)}{r - 1}$	B1	1																																	
6		<table border="1" style="margin-left: auto; margin-right: auto;"> <thead> <tr> <th>p</th> <th>$2^p - 1$</th> <th>Prime?</th> <th>n</th> </tr> </thead> <tbody> <tr> <td>1</td> <td>1</td> <td>No</td> <td></td> </tr> <tr> <td>2</td> <td>3</td> <td>Yes</td> <td>6</td> </tr> <tr> <td>3</td> <td>7</td> <td>Yes</td> <td>28</td> </tr> <tr> <td>4</td> <td>15</td> <td>No</td> <td></td> </tr> <tr> <td>5</td> <td>31</td> <td>Yes</td> <td>496</td> </tr> <tr> <td>7</td> <td>127</td> <td>Yes</td> <td>8128</td> </tr> <tr> <td>13</td> <td>8191</td> <td>Yes</td> <td>33550336</td> </tr> </tbody> </table>	p	$2^p - 1$	Prime?	n	1	1	No		2	3	Yes	6	3	7	Yes	28	4	15	No		5	31	Yes	496	7	127	Yes	8128	13	8191	Yes	33550336	M1 A1 A1	3	Extend table Correct table -1 each error, omission or extra
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