MEI Structured Mathematics

Practice Comprehension Question - 3

(Concepts for Advanced Mathematics, C2)

Choosing your Weights

Suppose you wish to weight objects that are an exact number of grams on a balance with the object on the right hand pan and the weights on the left hand pan.

It is possible to weigh any mass providing you have enough weights. Clearly if you had a weight for every possible mass then you can weigh all possible masses. But is it possible to weigh, say 8 different objects with less than 8 weights? Again, the answer is clearly yes because you can use two or more weights.

So if you have a weight of 1g and 2g you can weigh 1g and 2g and also 3g masses.

But you can do better than this. If the weights are 1g and 3g then you can weigh objects with these masses and a mass of 4g, but if you were also allowed to place the 1g on the right hand pan then you can weigh an object of 2g. So with the two weights you can weigh 1g, 2g, 3g and 4g masses.

Likewise, with the three weights 1g, 3g and 9g you can weigh objects of mass from 1g up to 13g.

To weigh an object of 5g for instance you would put the 9g on the left hand pan and the 1g and 3g together with the object on the right hand pan.

This is because 9 = 1 + 3 + 5.

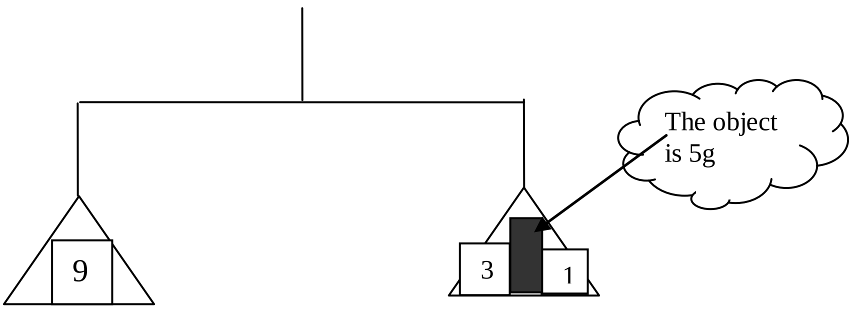

Fig. 1

Suppose now that you are allowed a fourth weight? What would it be?

The lightest object you have been unable to weigh so far is 14g. To weigh this you need an extra weight, Wg. If you put this new weight on the left hand pan and the object plus all the other weights on the right hand pan then:

$$W = 14 + 1 + 3 + 9 = 27$$

So the extra weight you will need is 27g.

You can make this procedure into a general formula for choosing further weights.

Call the total of all the weights you have already got ***T***g. So the next object to be weighed must be (***T*** + 1)g. Your new weight, ***W***g, must equal the sum of the object and all the previous weights.

$$\boldsymbol{W} = (\boldsymbol{T} + 1) + \boldsymbol{T}.$$
$$\Rightarrow \boldsymbol{W} = 2\boldsymbol{T} + 1$$

You will notice that the first four weights go up in powers of 3. They are the first four terms of a geometrical sequence 1, 3, 9, 27,….. Is this chance, or is there a connection between the powers of 3 and the formula $\boldsymbol{W} = 2\boldsymbol{T} + 1$?

The sum of the first ***n*** terms of the geometric sequence 1, 3, 3^2,…. is

$$\boldsymbol{S}_{\boldsymbol{n}} = \frac{3^{\boldsymbol{n}} - 1}{3 - 1} = \frac{1}{2}\left(3^{\boldsymbol{n}} - 1\right)$$

$\boldsymbol{S}_{\boldsymbol{n}}$ is the sum of all weights, or in the previous notation, $\boldsymbol{S}_{\boldsymbol{n}} = \boldsymbol{T}$
This gives $2\boldsymbol{T} + 1 = 3^{\boldsymbol{n}}$.
In other words the next weight to be chosen is $3^{\boldsymbol{n}}$g - the next term in the sequence.

Another interesting feature of this situation is that the choice of weights is efficient. The set allows you to weight as many different objects as possible with a given number of weights with no overlaps.
Take the case of the three weights 1g, 3g and 9g.

Using 1 weight only you can weigh 3 different objects.
Using 2 weights only you can weigh 3 different objects.
Using all 3 weights you can weigh one object.
But with one weight you can put one or both of the smaller weights on the other side.
So for 3g you can weigh 1 object
For 9g you can 3 objects.
Finally you can place two weights on the left and the third on the right, providing the one on the right does not exceed the sum on the left.
This gives you 2 further weights.
This gives a total of 13 weights.

These 13 are the only possible ways in which you can arrange the three weights, and they correspond to the 13 different objects. Each weight from 1g to 13g can be weighed and there will be no duplication.

To show that a set of weights is efficient you need to establish that two conditions are met

A You can weigh every object up to the maximum possible
B The number of possible objects is the same as the number of possible arrangements of the weights between the scale pans. This is equivalent to saying there is no duplication; no object can be weighed using two different arrangements of the weights.

The set 1, 3, 10 is not efficient because it fails condition A. It is not possible, for instance to weigh an object of mass 5g.
The set 1, 3, 8 is not efficient because there are 13 possible arrangements of 3 weights, but 1 + 3 + 8 = 12. In this case 4is duplicated. 3 + 1 = 4 and also 8 – 3 – 1 = 4.

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The argument that follows shows that the set of weights 1, 3, 9 is efficient.

The set 1, 3 is efficient because it fulfils both condition A and B.

A You can weigh all objects up to 4g as follows
$1 = 1, \quad 2 = 3 - 1, \quad 3 = 3. \quad 4 = 3 + 1$

B You can choose 1 weight: 1 or 3 — 2 arrangements
You can choose 2 weights: $3 - 1$, $3 + 1$ — 2 arrangements
Total 4 arrangements

Now include 9 in the set.

A You can weigh all objects up to 13g as follows

1, 2, 3, 4	as above
5, 6, 7, 8	as 9 – (1 or 2 or 3 or 4)
9	on its own
10 , 11, 12, 13	as 9 + (1 or 2 or 3 or 4)

B Part A also gives you all the possible arrangements of the weights, and these number 13 in all.

So the set 1, 3, 9 is efficient.

Now include 27 in the set.

A By a similar argument you can weigh all the objects up to 40g
B This also gives you all the possible rearrangements of the weights, and these number 40 in all.

So the set 1, 3, 9 and 27 is efficient.

By a similar process this proof can go on ***ad infinitum*** to show that the sets $\{1, 3, 3^2, 3^3, \ldots 3^{n-1}\}$ which are powers of 3 is efficient in weighing all objects up to $\frac{1}{2}\left(3^n - 1\right)$.

Questions:

1 **(i)** Line 12 asserts that with the three weights 1g, 3g and 9g you can weigh objects of mass from 1g up to 13g. Show how an object of 11g can be weighed. [1]

(ii) Line 21 asserts that with the extra weight of 27g you can weigh 14g. Up to what weight can you now weigh with the four weights? [2]

(iii) If you were told to choose 6 weights, what would they be to be efficient? [1]

2 Explain how the expression for S_n in line 31 can be obtained. [2]

3 Complete the proof that the set 1, 3, 9 and 27 is efficient as shown for the set 1, 3 and 9. [3]

4 Justify the last sentence - that the set $\{1, 3, 9, \ldots 3^{n-1}\}$ can weigh objects up to a mass of $\frac{1}{2}\left(3^n - 1\right)$. [3]

5 In the article the weighing was done with scale pans in which weights can be added to the right hand pan, where you place the object. In a different weighing arrangement the object is hung from a hook and any weights used must be on the left hand pan. This is illustrated in Fig. 2.

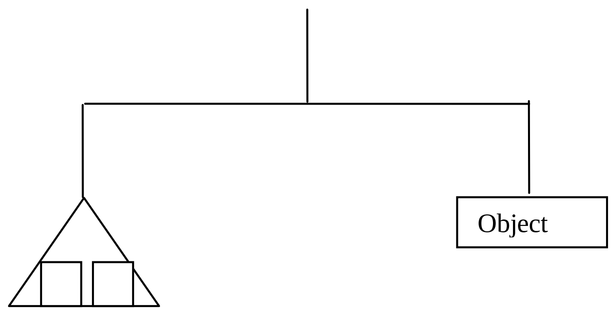

Fig. 2

(i) What sequence of weights would you now use? [2]

(ii) Use an argument similar to that in lines 61 - 75 to show that this set of weights is efficient. [4]

Answers.

1	**(i)**	9 and 3 on left, place 1 with object on right.	B1 **1**	
	(ii)	1 + 3 + 9 + 27 = 40	M1 A1 **2**	
	(iii)	1, 3, 9, 27, 81, 243	B1 **1**	
2		General formula for a GP with first term $\boldsymbol{a}$ and common ratio $\boldsymbol{r}$ is $S_n = a + ar + ar^2 + \ldots .ar^{n-1} = \frac{a(r^n - 1)}{r - 1}$ In this case $a = 1$ and $r = 3$	M1 A1 **2**	formula for GP given Values for $\boldsymbol{a}$ and $\boldsymbol{r}$ identified.
3		A You can weigh all objects up to 40g as follows 1 - 13 as above 14 - 26 27 – (1 - 13) 27 on its own 28 - 40 27 + (1 - 13) B Part A also gives you all the possible arrangements of the weights, and these number 40 in all.	M1 A1 B1 **3**	
4		Greatest weight is $1+3+9+\ldots\ldots .3^{n-1} = \frac{3^n - 1}{3-1} = \frac{1}{2}(3^n - 1)$	M1 A1 A1 **3**	
5	**(i)**	1, 2, 4, 8….	B2 **2**	
	(ii)	The set 1, 2 is efficient because A 1 = 1, 2 = 2, 3 = 1 + 2: You can weight all objects up to 3g B The possible arrangements are: Choose 2 weight 1, 2 2 arrangements Choose 2 weights 1, 2 1 arrangements Total 3 arrangements So the set 1, 2 is efficient. Now include 4 in the set A 1, 2, 3 as above 4 on its own 5, 6, 7 as 4 + (1, 2 or 3)	B2 B2 **4**	