MEI Structured Mathematics

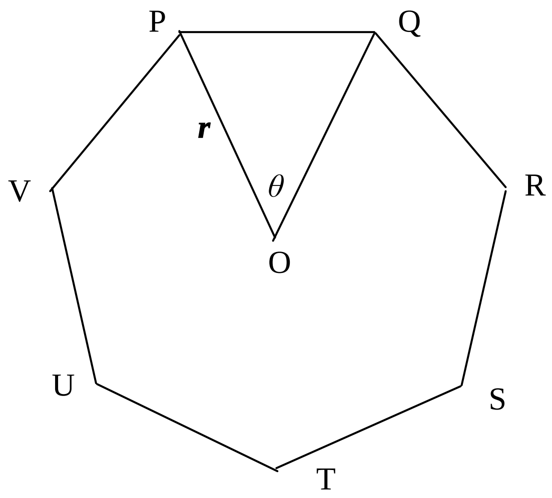
Practice Comprehension Question - 4

(Concepts for Advanced Mathematics, C2)

The 50p coin.

Feel a 50p coin. It has 7 edges but they are not straight lines; they are arcs of circles. What is the percentage increase in the amount of metal used to make the coin like that?

To answer this question, start by looking at the regular heptagon in Fig. 1.



. **Fig.1**

The angles POQ, QOR,.. are all equal and are therefore all equal to $\theta = \frac{360}{7}$.
Likewise, the lengths OP, OQ, OR …. are all the same and equal to the radius of the circle that circumscribes the regular heptagon. In this article this radius is denoted by $\boldsymbol{r}$.

To find the area of the heptagon, we first find the area of the triangle POQ.
There are several formulae for finding the area of a triangle when one angle is not a right angle. One of them is

$$\text{Area} = \frac{1}{2}\boldsymbol{ab}\sin\boldsymbol{C}$$

Applying this formula to the triangle POQ gives $\frac{1}{2}\boldsymbol{r}^2\sin\theta$, and it follows that the area of the heptagon is approximately $2.73641\boldsymbol{r}^2$.

Now for the 50p coin.

Each of the edges is not a straight line, but the arc of a circle.

The centre of this circle is the vertex opposite the edge.

So T is the centre of the arc PQ.

The region enclosed in the diagram is a sector with centre T and radius TP.

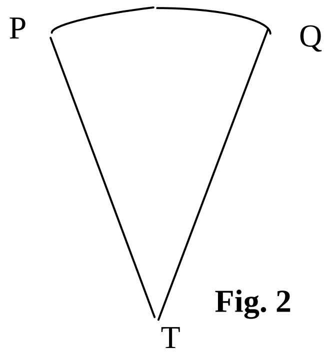


Fig. 2

In order to find the area of this sector we need the angle PTQ and the radius TP (= TQ).

Because O is the centre of the circumscribing circle of the heptagon,

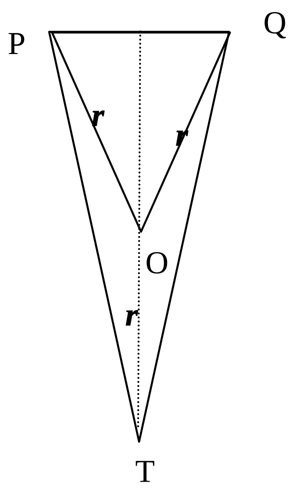


Fig. 3

Angle PTQ = $\frac{1}{2} \times$ Angle POQ = $\frac{1}{2} \times \frac{360}{7} = \frac{180}{7}$.

Therefore Angle PTO = $\frac{1}{2} \times$ Angle PTQ = $\frac{1}{2} \times \frac{180}{7} = \frac{90}{7}$.

Triangle POT is isosceles, with PO = OT = $\boldsymbol{r}$.

Hence PT $= 2\boldsymbol{r}\cos\frac{90}{7}$

So the area of the sector is

$$= \pi \mathrm{PT}^2 \times \frac{\text{angle PTQ}}{360}$$

$$= \pi\left(2\boldsymbol{r}\cos\frac{90}{7}\right)^2 \times \frac{180/7}{360}$$

$$= \frac{2\pi}{7}\boldsymbol{r}^2\cos^2\frac{90}{7}$$

We are now in a position to find the extra metal in the 50p coin by considering the shaded area in Fig. 4. This area is the difference in the area of the sector PTQ and the triangle PTQ.

The area of triangle PTQ $= \frac{1}{2}\boldsymbol{ab}\sin\boldsymbol{C}$

$$= \frac{1}{2}\text{PT}^2 \sin\frac{180}{7}$$

$$= \frac{1}{2}\left(2\boldsymbol{r}\cos\frac{90}{7}\right)^2 \sin\frac{180}{7}$$

$$= 2\boldsymbol{r}^2\cos^2\frac{90}{7}\sin\frac{180}{7}$$

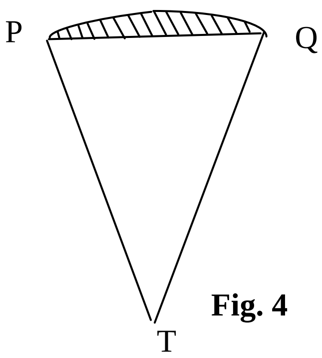


Fig. 4

Area = Area of sector PTQ − Area of Triangle PTQ 35

$$= \left(\boldsymbol{r}^2\cos^2\frac{90}{7}\right)\left(\frac{2\pi}{7} - 2\sin\frac{180}{7}\right)$$

$$= 0.028353\boldsymbol{r}^2$$

So the area of the 50p coin is $2.7364\boldsymbol{r}^2 + 7\times 0.028353\boldsymbol{r}^2 = 2.934871\boldsymbol{r}^2$.

The percentage increase in the amount of metal needed is therefore approximately 7.25%.

Questions:

1 Prove the formula for the area of a triangle Area $= \frac{1}{2}\boldsymbol{ab}\sin\boldsymbol{C}$, given in line 10. [2]

2 Show that the area of the regular heptagon is $2.73641\boldsymbol{r}^2$, as stated in line 12. [3]

3 State the circle theorem being used in line 22. [1]

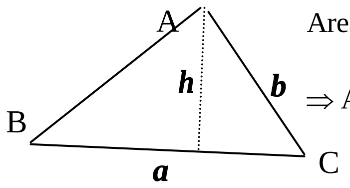
4 Derive the formula in line 25: PT $= 2\boldsymbol{r}\cos\frac{90}{7}$ [2]

5 Justify the last statement that the increase in metal is 7.25% [3]

6 In 1997 the size of the 50p coin was reduced in size while remaining similar in shape. Explain why the final answer remains unaltered. [1]

7 What would be the percentage increase if the 50p coin had been a pentagon? [6]

Answers.

1		 Area = $\frac{1}{2}\boldsymbol{ah}$ where $\boldsymbol{h} = \boldsymbol{b}\sin\boldsymbol{C}$ $\Rightarrow$ Area = $\frac{1}{2}\boldsymbol{ab}\sin\boldsymbol{C}$	M1 A1 **2**	
2		Area OPQ = $\frac{1}{2}\boldsymbol{r}^2\sin\theta$. $\Rightarrow$ Area of heptagon = $7\times\frac{1}{2}\boldsymbol{r}^2\sin\theta = \frac{7}{2}\boldsymbol{r}^2\sin 51.43$ $= 2.73641\boldsymbol{r}^2$	M1 M1 A1 **3**	
3		Angle at centre = 2 × angle at circumference.	B1 **1**	
4		$\frac{1}{2}$ PT $= \boldsymbol{r}\sin \text{PTO}$ $\Rightarrow$ PT $= 2\boldsymbol{r}\sin\frac{90}{7}$	M1 A1 **2**	
5		Area of 50p coin = $2.93487\boldsymbol{r}^2$. Area of heptagon = $2.73641\boldsymbol{r}^2$ $\Rightarrow$ Percentage increase = $\frac{2.93487\boldsymbol{r}^2 - 2.73641\boldsymbol{r}^2}{2.73641\boldsymbol{r}^2}\times 100$ $= \frac{0.19846\boldsymbol{r}^2}{2.73641\boldsymbol{r}^2}\times 100$ $= 7.25$	M1 A1 A1 **3**	
6		Final answer is independent of $\boldsymbol{r}$.	B1 **1**	

7		Angle POQ = $\frac{360}{5} = 72$	B1	
		$\Rightarrow$ Area Triangle POQ = $\frac{1}{2}\boldsymbol{r}^2 \sin 72 = 0.47553\boldsymbol{r}^2$	M1	
		$\Rightarrow$ Area pentagon = $5 \times 0.47553\boldsymbol{r}^2 = 2.3776\boldsymbol{r}^2$	A1	
		Angle PTQ = $\frac{1}{2} \times 72 = 36 \Rightarrow$ Angle PTO = $\frac{1}{2} \times 36 = 18$	M1	
		$\Rightarrow$ PT $= 2\boldsymbol{r}\cos 18$		
		$\Rightarrow$ Area Triangle PQT = $\frac{1}{2}\left(2\boldsymbol{r}\cos 18\right)^2 \sin 36 = 1.0633\boldsymbol{r}^2$		
		and Area Sector PTQ = $\pi\left(2\boldsymbol{r}\cos 18\right)^2 \frac{36}{360} = 1.13663\boldsymbol{r}^2$	A1	
		$\Rightarrow$ Area of extra bits = $5\left(1.13663\boldsymbol{r}^2 - 1.0633\boldsymbol{r}^2\right) = 0.3666\boldsymbol{r}^2$		
		Percentage increase = $\frac{0.3666\boldsymbol{r}^2}{2.3776\boldsymbol{r}^2} \times 100$ $= 15.4\%$	A1 **6**	